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Domain wall junctions in supersymmetric field theories in $D = 4$

Soonkeon Nam* **and Kasper Olsen**

Department of Physics, Harvard University, Cambridge, MA 02138, USA
E-mail: nam@pauli.harvard.edu, kolsen@feynman.harvard.edu

Abstract: We study the possible BPS domain wall junction configurations for general polynomial superpotentials of $\mathcal{N} = 1$ supersymmetric Wess-Zumino models in $D = 4$. We scan the parameter space of the superpotential and find different possible BPS states for different values of the deformation parameters and present our results graphically. In this way we obtain a complete picture of the possible BPS domain wall junctions for the models we consider. We comment on the domain walls in F/M/IIA theories obtained from the Calabi-Yau fourfolds with isolated singularities and a background flux.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Field Theories in Lower Dimensions.

*Permanent Address : Department of Physics, Kyung Hee University; Seoul, 130-701, Korea, and APCTP Yoksam-dong, Kangnam-Gu; Seoul, 135-080, Korea, E-mail: nam@string.kyunghee.ac.kr

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1. Introduction

Domain walls arise in scalar field theories as solutions connecting two isolated vacua which are degenerate. Physical examples can range from a system of liquid crystals [1] to defects in cosmological models [2]. A simple way to obtain a theory with degenerate vacua is to consider a supersymmetric field theory. In this case supersymmetry guarantees the positivity of the scalar potential $V(\Phi)$, which can be written in terms of superpotential $W(\Phi)$, i.e. $V(\Phi) \sim |\partial W(\Phi)/\partial\Phi|^2$. The location of the minima of the potential are at the critical points $\Phi = \Phi_k$ of the superpotential, such that $W'(\Phi_k) = 0$.

Starting from the simplest model of a single scalar field theory with a potential in $1+1$ dimensions, which allows a single type of domain wall between each of the critical points, things get more complicated when we consider theories with multiple scalar fields and multiple critical points. In cases where there are more than two degenerate vacua, one might consider any pair of vacua and try to connect them with a domain wall (or soliton in $1+1$ dimension). However, this simple-minded construction cannot always be realized since there might not always be a BPS solution connecting two given vacua. This can be exemplified by the Wess-Zumino(W-Z) model with the following quintic superpotential: $W(\Phi) = \Phi^5/5 - \Phi^2/2$, which has four critical points, one at $\Phi = 0$ and three others at vertices of an equilateral triangle. In this theory the domain wall which interpolates between $\Phi = 0$ and any one of the corners exists, but direct connection of two of the vertices does not exist [3]. Therefore such a superpotential only allows for three BPS states and not six as one might have expected. (For this particular example, one can actually see from surface plot of the potential $V(\Phi)$ that there is no BPS path between the vertices of the triangle.)

In $1+1$ dimensions these interpolating BPS solutions are just kinks or solitons. Integrability conditions for different soliton solutions in $1+1$ dimensions, interpolating different pairs of critical points were studied in ref. [3], where a soliton which saturates the Bogomol'nyi bound can best be described as a straight line connecting the critical points in the superpotential space, i.e. the W-plane. In fact a very extensive classification program of integrable models was carried out in $(1+1)$-dimensional theories with $\mathcal{N}=2$ supersymmetry in ref. [4]. Some of the results there can be used in higher-dimensional theories with domain walls because domain walls essentially have one space dimensional dependence, which is along the direction separating two domains. One new feature that appears when we have more than one spatial dimension is that we can now have intersections or junctions of domain walls [5]. We can ask a similar question for the existence of a BPS state between critical points each time we encounter a superpotential, and perform an analysis as was done extensively in ref. [6]. However, it would be desirable to have a more global view in the parameter space (i.e. the space of deformations of the superpotential) so that we can easily follow the behavior of certain BPS states which are created or destroyed as we move around in this parameter space.

In this paper, we will consider domain walls and their junctions in $\mathcal{N}=1$ supersymmetric field theories in four dimensions and we analyze under which circumstances certain classes of junctions can appear or not. For an appropriate choice of superpotential, such domain walls have been shown to arise in the W-Z model and also in SU(N) SUSY QCD for which the W-Z model is an effective low-energy theory. Furthermore, it has been shown that the W-Z model (at least for a $\mathbb{Z}_3$ symmetric configuration of three critical points) admits solutions preserving only 1/4 of supersymmetry [7, 8, 9], which were interpreted as junctions of three domain walls. More general BPS and non-BPS junctions of the W-Z model with a $\mathbb{Z}_k$ symmetric configuration of critical points where discussed in [10]. Recently non-perturbative junctions of domain wall solutions were also extensively studied in SUSY QCD [11], and in the brane world scenarios [12, 13, 14], where gravitating domain wall junctions were considered.

Another important motivation to study this subject comes from the recent discussions of the vacuum and soliton structure of supersymmetric theories in the context of string theory compactifications [15, 16]. Consider compactification of type-II, M-, or F-theory on some singular non-compact Calabi-Yau n manifold with some background flux of Ramond-Ramond field, say G. (For F-theory, we need elliptically fibered Calabi-Yau manifold, and in addition we need both NS and RR fluxes.)

Non-vanishing RR flux is needed to cancel the tadpole anomaly [17], while taking a singular limit of a Calabi-Yau manifold leads to a decoupling of gravity in the effective field theory in the lower dimension [18]. Domain walls are identified with D-branes (or M-branes for M-theory) wrapped on supersymmetric cycles and in crossing such a brane the flux (of the appropriate field) jumps, so the different values of the flux correspond to different vacua. For supersymmetric vacua certain conditions has to be imposed on G [19]. These constraints can be realized by interpreting G as giving rise to an effective superpotential of the lower-dimensional theory which is of the form

$$W = \int A \wedge G \,, \tag{1.1}$$

where A is either the holomorphic n-form Ω or some appropriate power of the Kähler potential $\mathcal{K}$.[1] For compactification of type-II, M-theory or F-theory on *singular* Calabi-Yau manifolds this analysis leads in certain cases to an identification of the corresponding low-dimensional theories as specific non-trivial conformal field theories, depending on the singularity in question. As an example, it was shown [16] that type IIA compactified on a Calabi-Yau four-fold with A_n singularity gives an $\mathcal{N} = 2$ Kazama-Suzuki model [21] in two dimensions.

In this paper, we will concentrate on W-Z models in four dimensions (with four supercharges), though much of the analysis can be applied in three and two dimensions as well. We analyze the appearance of BPS domain walls and junctions for massive deformations away from the conformal point. In section 2, we review the possibility of central charges of the $\mathcal{N} = 1$ superalgebra in four dimensions and their interpretation in terms of domain wall and junction charges and also the BPS condition for the domain walls and their junctions. In section 3 we review the derivation of W-Z models in $D = 2, 3$ from type-IIA or M-theory and discuss some relations between the geometry of the Calabi-Yau manifold and the solutions of the BPS equation in lower dimensions. We also comment about generating superpotentials in F-theory. In section 4, we collect the rules for the counting of BPS states, which are used in section 5 in studying massive deformations of the W-Z model with a general quintic superpotential. Finally, section 6 contains our discussions.

2. Supersymmetry algebra and the BPS condition

We start by recalling the structure of the $\mathcal{N} = 1$ supersymmetry algebra in $3 + 1$ dimensions and how the possibility of domain walls and junctions of domains walls can be analyzed directly from this algebra. (For further discussions of the $\mathcal{N} = 1$ algebra in $D = 4$, see [7, 8, 22, 23].)

[1] Note that this is related to the theory of *calibrations*: A is the calibration and for $A = \Omega$ these potentials are related to lagrangian submanifolds and give rise to chiral superfields, while if $A = \mathcal{K}^p$ they are related to holomorphic curves and lead to "twisted" chiral superfields. [20]

The $\mathcal{N} = 1$ supersymmetry algebra in $D = 4$ allows central charges which correspond to tensions of BPS domain walls and junctions of them [24, 22]:

$$\{Q_\alpha, Q_\beta\} = 2i(\sigma^k \overline{\sigma}^0)_\alpha{}^\gamma \epsilon_{\gamma\beta} Z_k\,, \qquad \{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^\mu_{\alpha\dot\alpha} P_\mu + \sigma^k_{\alpha\dot\alpha} Y_k)\,, \tag{2.1}$$

where $k = 1, 2, 3$ and $\mu = 0, \ldots, 3$. The Z_k (which are complex charges) have an interpretation as domain wall charges and Y_k (which are real charges) as the junction energy, which can be either positive or negative [22].

The relations between the superpotential and the central charges are given by

$$Z_k = 2\int d^3x \partial_k W^*(\phi^*)\,, \tag{2.2}$$

$$Y_k = i\epsilon^{knm} \int d^3x K_{i\bar{j}} \partial_n (\phi^{*j} \partial_m \phi^i)\,, \tag{2.3}$$

where ϕ is the scalar component of the chiral superfield, and the Kähler metric is derived from the Kähler potential K via $K_{i\bar{j}} = \partial^2 K / \partial\phi^i \partial\phi^{*j}$. The central charges Z_k depend only on the difference between the values of the superpotential at spatial infinity. If we have a single domain wall — which is only non trivial in one dimension — then Y_k vanishes for all k and Z_j is non vanishing (for some j) since the Z_j central charge depends on the spatial derivative in the x_j direction. For junctions of domain walls to be possible, one first of all need more than a single (real) scalar field, since else Y_k will vanish identically. Furthermore, as one can clearly see, Y_k is non vanishing only when the field configuration at infinity is non trivial in two dimensions. If we have two spatial dependences as for a domain wall junction solution, then we will generically have two of the Z_k's non vanishing. When the Kähler metric is trivial, Y_k is just a surface integral at the infinity.

When we start from a $\mathcal{N} = 1$ theory in $D = 4$ we originally have four supercharges. Domain wall configurations with non-zero Z_k's and vanishing Y_k has two conserved supercharges, thus are 1/2 BPS states, whereas when there is non-zero Y_k there is only one combination of the four supercharges which can survive. This leads to a 1/4 BPS state.

The BPS equation for a single static domain wall of the W-Z model (dimensionally reduced to two dimensions) is given by:

$$\partial_x \phi = e^{i\alpha} \overline{W'}\,, \tag{2.4}$$

where the prime denotes derivative with respect to $\phi = \phi(x)$, and x is a coordinate and α is an arbitrary phase. A domain wall solution of mass M saturates the bound $M \geq |T|$, where T is the topological charge associated with the wall and has $\alpha = \arg T$. The BPS equation for a domain wall junction can be derived in higher space dimension as in [7] and is completely analogous to eq. (2.4). In particular, if we suppress spatial dependences other than two of them, say x and y, then the BPS equation becomes

$$(\partial_x - i\partial_y)\phi = e^{i\alpha} \overline{W'}\,. \tag{2.5}$$

The BPS solution saturates the bound $M \geq |T| + Q$, where Q is the junctions charge. When there is only on spatial dependence, e.g. $\partial_y \phi(z) = 0$, this reduces to eq. (2.4). Note that this junction is an object in a three-dimensional theory and not a two-dimensional theory as the one discussed in [7].

3. Wess-Zumino models from Calabi-Yau compactifications

In this section we discuss how W-Z models naturally appear in CY-compactifications with background fluxes. Though this section essentially does not contain any new results, it provides the motivation for studying domain walls of W-Z models in the following sections. We will comment on the applications in section 6.

The W-Z model we will consider will be a field theory in $D = 4$ dimensions with $\mathcal{N} = 1$ supersymmetry. It has a superpotential $W(\Phi)$ which is of the form

$$W'(\Phi) = C \prod_{i=1}^{m} (\Phi - \lambda_i) \,, \tag{3.1}$$

where C is a constant and λ_i, $i = 1, \ldots, m$ are the locations of the critical points in the field space. The $\mathbb{Z}_m$ symmetric case, for $\lambda_i = |\lambda_0| e^{2\pi i/m}$, $(i = 1, \ldots, m)$, were considered in connection to the $\mathcal{N} = 1$ supersymmetric SU(N) YM theory in the large-N limit [11]. Although it is believed that W-Z models for $m > 3$ will flow to trivial IR theories for $D \geq 3$, it can become relevant as a perturbations to certain fixed points. Furthermore one can have non-trivial brane configurations realizing these higher order potentials [25]. We will comment on this in the last section.

As discussed in [16] Wess-Zumino models can arise in Calabi-Yau compactifications of M/type-IIA theories. The locations of the isolated singularities correspond to the locations of the critical points. In the case of an A_k singularity, the local geometry of the Calabi-Yau n-fold is described by an equation of the form

$$-P_m(z_1) + z_2^2 + \cdots + z_{n+1}^2 = 0 \tag{3.2}$$

with $P_m(z_1)$ a generic polynomial of degree $m = k + 1$ in z_1:

$$P_m(z_1) = \prod_i (z_1 - a_i) \,. \tag{3.3}$$

In the above a_i are the locations of the singularities. When we fix z_1 we can regard $|P_m(z_1)|$ as the radius of the $n-1$ sphere which is the non-trivial cycle of the manifold. Since the mass of the brane wrapping around the singularity will be proportional to the volume of the sphere, and this mass will give also the tension of the domain wall in the lower effective field theory, we get the following fact that the superpotential W is related to P_m through

$$dW = P_m^{\frac{n-2}{2}} dz_1 \,, \tag{3.4}$$

where the right hand side is basically the volume of the sphere. So, for Calabi-Yau fourfold compactifications we recover the W-Z superpotentials.

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Instead of going directly to the case of a quintic superpotential, we would like to discuss some general features of the solutions of the BPS equation corresponding to compactification on a general Calabi-Yau n-fold. BPS states are in this case identified with wrapped n-branes in a Calabi-Yau n-fold near an isolated singularity [16]. The kind of singularities we will be looking at are the A_k singularities, which are described by the eq. (3.2). Any such Calabi-Yau manifold has a unique n-form Ω which determines the volume of a cycle C. The condition for a cycle to be minimal is that its volume saturates the inequality

$$V = \int_C |\Omega| \geq \left| \int_C \Omega \right| . \tag{3.5}$$

Now, the problem of minimizing this volume can be mapped to a problem in the complex z_1-plane as follows. One considers the n-cycle to be an n-sphere S^n, which is locally of the form $S^{n-1} \times S^1$, i.e. as an S^{n-1}-sphere fibered over a real curve in the z_1-plane. The local volume of this S^{n-1}-sphere is determined by $z_2^2 + \cdots + z_{n+1}^2 = P_m(z_1)$ and so vanishes at the roots of $P(z_1)$, which are identified with the critical points of the superpotential $W(z_1)$. With this choice of local coordinates on the singular Calabi-Yau, the expression for the holomorphic n-form is

$$\Omega = \frac{dz_1 \cdots dz_n}{z_{n+1}} = \frac{i dz_1 \cdots dz_n}{\sqrt{z_1^2 + \cdots + z_n^2 - P_m(z_1)}} . \tag{3.6}$$

The condition for a cycle to be supersymmetric is that the image of the path is a straight line in the flat W-plane, where W is defined through the relation in eq. (3.4). This comes from minimizing the l.h.s. of (3.5) with the expression (3.6) for Ω. The BPS condition is then:

$$W(z(t)) = \int_{z_0}^{z(t)} P^{\frac{n-2}{2}} dz = \alpha t \, , \tag{3.7}$$

where t parametrizes the curve connecting the two critical points. To obtain the BPS states one should therefore solve the first-order differential equation:

$$P^{\frac{n-2}{2}} \frac{dz}{dt} = \alpha \tag{3.8}$$

with the boundary conditions that $z(t)$ should begin and end at the roots of $P(z)$ (or rather of $dW(z)$). Near a root, which we take to be at $z = 0$, one is solving an equation of the form

$$\frac{dz}{dt} = \frac{\alpha}{z^{\frac{n-2}{2}}} , \tag{3.9}$$

for which the solution is

$$z = \left(\frac{n}{2} \alpha t \right)^{2/n} . \tag{3.10}$$

In the case of a Calabi-Yau four-fold we see that there are four solutions for any α and that the corresponding curves intersect at an angle of 90°.

	M-theory	IIA-theory	F-theory
Vacuum:	$\mathbb{R}^3 \times Y$	$\mathbb{R}^2 \times Y$	$\mathbb{R}^4 \times Y$
	$G = dC$	$G = dC$	$\mathrm{D7} = \mathbb{R}^4 \times L$
	$G \in H^4(Y;\mathbb{Z}) = \Gamma^*$	$G \in H^4(Y;\mathbb{Z}) = \Gamma^*$	$F \in H^2(L;\mathbb{Z}) = \Gamma^*$
Domain wall:	$\mathrm{M5} = \mathbb{R}^2 \times S$	$\mathrm{D4} = \mathbb{R} \times S$	$\mathrm{D5} = \mathbb{R}^3 \times V$
	$[S] \in H_4(Y;\mathbb{Z}) = \Gamma$	$[S] \in H_4(Y;\mathbb{Z}) = \Gamma$	$[\partial V] \in H_2(L;\mathbb{Z}) = \Gamma$

Table 1: M/IIA/F-theory on Calabi-Yau four-fold Y.

Now we will discuss how to construct domain walls in such Calabi-Yau compactifications and we will follow the discussion in [16]. For more details, see also [26, 27]. Consider compactification of M-theory on some Calabi-Yau four-manifold Y with some background flux of the three-form potential C which couples to the membranes (see table 1, in which we summarize the construction of vacua and domain walls in compactification of M/IIA/F-theory with background fluxes as in [16]). These C-field are classified by a class $\xi \in H^4(Y;\mathbb{Z})$, which in turn defines a lattice $\Gamma^* = H^4(Y;\mathbb{Z})$. This set of data specifies a choice of vacuum. Now, to make a domain wall in $\mathbb{R}^3$, one considers a fivebrane with worldvolume $\mathbb{R}^2 \times S$, where S is a four-cycle. Such four-cycles are classified by $H_4(Y;\mathbb{Z})$, which defines a lattice Γ dual to Γ^*. So when crossing such a domain wall ξ changes — and the possible values of ξ are classified by Γ^*/Γ. The effective superpotential is obtained as follows. When crossing a domain wall, the change in the superpotential is

$$\Delta W = \frac{1}{2\pi}\int_X \Omega \wedge \Delta G\,, \tag{3.11}$$

where $G = dC$. Here $X = \mathbb{R}^3 \times Y$. This superpotential will then account for the restrictions on G (which are implied by having vacua with supersymmetry) mentioned in the introduction.

Compactifying type IIA on Y, we have $X = \mathbb{R}^2 \times Y$. And to specify a vacuum we should also specify the topological class of the G-field, which is now the RR four-form field and takes values in $H^4(Y;\mathbb{Z})$. To make a domain wall, one now has four-branes with worldvolume $\mathbb{R} \times S$. Again the possible four-cycles S are classified by $H_4(Y;\mathbb{Z})$ and therefore ξ takes values in Γ^*/Γ. Compactification of type IIA on Calabi-Yau four-fold Y, with A_k singularity, will then give an effective two-dimensional theory with superpotential determined by (3.4) for $n = 4$. This is precisely the superpotential discussed in the following sections, and here we can of course have domain walls between different vacua. But we will not have junctions.

The story for F-theory [28] compactifications is slightly different. First of all, for F-theory compactification on $\mathbb{R}^4 \times Y$ we need Y to be an elliptically fibered four-manifold. The flux Φ discussed in [16] now has contributions from space-filling threebranes and not membranes as in the compactification of M-theory. The analog of the G-field now becomes both NS and RR three-form fields, $H^{\rm NS}$ and $H^{\rm R}$, from the type-IIB theory. F-theory on $\mathbb{R}^4 \times Y$ can be described as type IIB with certain (p,q)-sevenbranes on a locus $L \subset B$, where B is the base of the elliptic fibration. However, in this situation, one can find a simpler description: This F-theory compactification with singularity can be reinterpreted as type IIB with a D7-brane with worldvolume $\mathbb{R}^4 \times L$, where L is a complex (singular) surface inside $\mathbb{C}^3$ (see table 1). This specifies a choice of vacuum. One has a U(1)-gauge field on the D7-brane and so this vacuum is characterized by the first Chern class, or an element ξ of the lattice $\Gamma^* = H^2(L;\mathbb{Z})$.

How do we construct domain walls? Take a D5-brane, which can end on the D7-brane, with worldvolume $\mathbb{R}^3 \times V$, where V is a three-manifold whose boundary should be in L (since the D5-brane ends on the D7-brane). This boundary defines a topological class $[\partial V] \in H_2(L;\mathbb{Z})$, i.e. in the lattice $\Gamma = H^2(L;\mathbb{Z})$ dual to Γ^*. Crossing the domain wall, the Chern class changes by the amount $[\partial V]$. Again ξ takes values in Γ^*/Γ. We also need to specify the local geometry of Y. For elliptic four-fold singularity one has the description

$$y^2 = x^3 + 3ax^2 + H(z_1, z_2, z_3)\,, \tag{3.12}$$

where H is a polynomial in z_1, z_2, z_3. The equation for L then becomes simply $H = 0$ and to describe an A_k-singularity one should then choose:

$$H = z_1^{k+1} + z_2^2 + z_3^2\,. \tag{3.13}$$

It would be desirable to have an explicit computation of the superpotential in F-theory generated by the inclusion of H-flux and with $A-D-E$-type singularities. For that one could start with type IIB on Calabi-Yau three-fold as in [29], where $W = \int \Omega \wedge (\tau H^{\rm NS} + H^{\rm R})$, and then lift this construction to F-theory.

4. Rules for the construction

Now we will discuss the rules for finding the number of BPS states for different values of the perturbation parameters, which translates to varying the positions of the critical points.

(1) *What are we constructing?*

From the BPS equation one can easily show that the BPS solution trajectories are straight lines between critical points in the W-plane [4]. However the inverse image of a certain straight line — connecting, say $W(i)$ and $W(j)$ — might not lift back to the field space as a curve connection the vacua and thus does not correspond to a BPS solution. To count the number of actual solutions connecting i and j, one starts with the "wavefront"(or sphere) of all possible solutions emanating from i with fixed values of W, denoted by Δ_i, and the same for the critical point j. The number of solutions is then exactly the number of points at which Δ_i and Δ_j intersect [4] (note that the intersection number depends on a choice of orientation and what we really are computing is a weighted sum [30]). This defines the intersection number $\mu_{ij} = \Delta_i \circ \Delta_j$ as a quantity which is invariant under small perturbations of the superpotential since it is integer. However, as we vary the superpotential the critical points will move around in the W-plane and when a third root k crosses the straight line connecting i and j the number of BPS solutions connecting i and j can obviously change. Precisely how this number changes can be derived using the Picard-Lefschetz theorem and is given by [4]

$$\mu'_{ij} = \mu_{ij} \pm \mu_{ik} \cdot \mu_{kj} \,. \tag{4.1}$$

Here the $\pm$ sign depends on the ordering of ikj in the triangle defined by the three roots before k was crossing the line between i and j. (Physically, this change in the intersection number, as one root crosses the line between two other roots, can be understood in terms of the Hanany-Witten effect [31, 32].) In principle on can determine these intersection numbers by solving the so-called tt* equation [6, 4] for a fixed choice of superpotential. But in our case we vary the parameters in the superpotential and it is more straightforward to look at conditions on masses of BPS solutions (and phases of the topological charges) to determine which kind of junctions exist or not. So we are in a certain sense trying to give a unified description of the cases considered in ref. [6].

So, it would be nice to have the form of μ_{ij}'s as functions of the parameters of the theory. Since it takes integer values, it is stable under small perturbations and changes only by an integer, and the best way to represent it would be to find the boundaries in the parameter space where the jump in the values happens. (This is called a *separatrix curve.*) Then we can specify the values of μ_{ij}'s inside each domain separated by the boundaries in the parameter space graphically. In the case of W-Z models we have $|\mu_{ij}| = 0$ or 1. Crossing a boundary induces a change in the number of BPS state of ± 1. So the graphical representation will be as follows. We will denote the critical points as dots. Then we will link the critical points i and j by a solid line if $|\mu_{ij}| = 1$. We will not link them if μ_{ij} vanishes. There will be at least one line coming from each critical point. (The connectivity is quite analogous to Dynkin diagrams.) So, if there are k critical points, there will be a maximum of $k(k-1)/2$ BPS states and a minimum of $k-1$ BPS states since all critical points can be connected through a sequence of BPS solutions [33].

(2) *What determines the separatrix equation?*

Observe that the topological charge associated with two critical points i and j is

$$T_{ij} = 2e^{i \arg(W(z_j)-W(z_i))}|W(z_j) - W(z_i)| \tag{4.2}$$

and so is a complex number. The mass M of a domain wall is bounded by the absolute value of the topological charge T:

$$M \geq |T| \,, \tag{4.3}$$

and is saturated by a solution of the BPS equation. Now consider a situation where i and j and also j and k are connected by a domain wall solution with BPS masses M_{ij}, M_{jk} and topological charges T_{ij}, T_{jk}. Let us consider the possibility of a BPS object between i and k. The possible BPS mass of such a solution is always bounded by the following simple inequality:

$$M_{ik} = |T_{ij} + T_{jk}| \leq |T_{ij}| + |T_{jk}| = M_{ij} + M_{jk}\,. \tag{4.4}$$

The inequality is saturated only when the phases of T_{ij} and T_{jk} are the same. When the equality (4.4) is saturated, such that $M_{ik} = M_{ij} + M_{jk}$, then the domain wall with mass M_{ik} decays into the two other domain walls. Since the phase of the topological charge comes from the argument of the difference of the superpotential, we can calculate the boundaries in the deformation parameter space where different solitons are created or destroyed as we change the parameters. Each such boundary is determined by three critical points and determines whether a solution between a particular pair of them becomes unstable or not. The entire parameter space will therefore be divided into many different domains and each domain will have the same number of possible BPS solutions.

(3) To map the entire parameter space we pick a point in the space where the BPS configuration is easily determined. As we move across a boundary a certain state can be created (if it was not there) or destroyed (if it was there). This technique will be applied in the next section where we find the separatrix curves for a general quintic superpotential.

5. Finding the BPS configurations

5.1 Quartic superpotential

The simplest non-trivial superpotential is of course one with two critical points. This allows a single BPS state and hence a single domain wall. Next would be one which has three critical points. In this case of $k = 3$ roots, and actually for all $k \geq 3$, one can argue that *any* pair of roots can be connected through a sequence of domain wall solutions [33]. By rescaling and fixing the value of the field we can fix two of the critical points to be, say at $z_1 = -1$ and $z_2 = 1$. The third critical point can be at an arbitrary point in the complex plane, say at $z_3 = \mu$ (this case is discussed in detail in [5]). When μ is a real number, $\mu > 1$, the critical points in the W-plane will be colinear and the only straight line connecting z_1 with z_3 will be through z_2. So there can only be two types of domain walls. The same conclusion — i.e. that there are only two BPS states — can be drawn when $|\mu| < 1$ for real μ, and also for $\mu < -1$. Let us now see what happens when we move away from the real line, holding fixed z_1 and z_2, for the case of $-1 < \mu < 1$. As $\mu = \mu_1 + i\mu_2$ (μ_1, μ_2 are real numbers) moves away from the real line, the number of BPS states stays the same until we reach a boundary in the complex μ plane where a new BPS state appears, arising from the domain wall between z_1 and z_2. This boundary is defined by the condition that the phases of the topological charges T_{13} and T_{32} are the same [5]. When the phases are the same then the inequality of the masses saturate and we have $M_{12} = M_{13} + M_{32}$ (M_{ij} is the mass of the soliton connecting the roots z_i and z_j). Similar boundaries can be found for the initial cases of $|\,\mathrm{Re}(\mu)| > 1$, determined by the equality of masses of M_{12} and M_{23} or M_{31} and M_{12}.

There is a reflection symmetry of the boundaries in the real line. These three boundaries together form the separatrix curve and the equation can be written down as the following condition on the real and imaginary parts of μ:

$$3\mu_1^4 + 2\mu_1^2\mu_2^2 - \mu_2^4 - 6\mu_1^2 - 6\mu_2^2 + 3 = 0\,. \tag{5.1}$$

Note that this equation *does not distinguish* which BPS state melts away as we cross a boundary. Different branches of eq. (5.1) will correspond to one of the boundaries which we discussed above, obtained from the relations between the possible masses of domain walls. So if we put $F_{ijk} \equiv M_{ij} + M_{jk} - M_{ik}$, the separatrix equation will be equivalent to $F_{123}F_{132}F_{312} = 0$, after some algebraic manipulations. This observation will be quite crucial in identifying various BPS states in the cases with more than three critical points. The real line will appear as a solution of the separatrix equation, but it will be a line of marginal stability, so the number of BPS states do not change as we cross the real line. The connectivity of the roots for the quartic superpotential is therefore very simple: either any root is connected to any other root (for a total of three BPS states), or two of the roots are not directly connected (for a total of two BPS states). This result is given in ref. [5, figure 4]. The connectivity signals a *possible* BPS state. It can be occupied or be vacant. Now when the occupied BPS states are such that we have an enclosed domain, then we have a junction of the three domain walls and a 1/4 BPS state. If they do not enclose a separate domain, say just two of the edges of a triangle, then we have two BPS domain walls which never join and the whole configuration will be 1/2 BPS. So when the positions of the critical points are more or less colinear (in W-space) domain wall junctions do not develop. This can be used in the cases with more than three critical points, where the positions of three particular ones will more or less follow the pattern described above, although the very existence of the other critical points do interfere with the detailed shape of the separatrix curves.

5.2 Quintic superpotential

Next we analyze the $D = 4$ W-Z model with a general quintic superpotential,

$$W = z^5 + \sum_{i=1}^{4} \alpha_i z^i\,, \tag{5.2}$$

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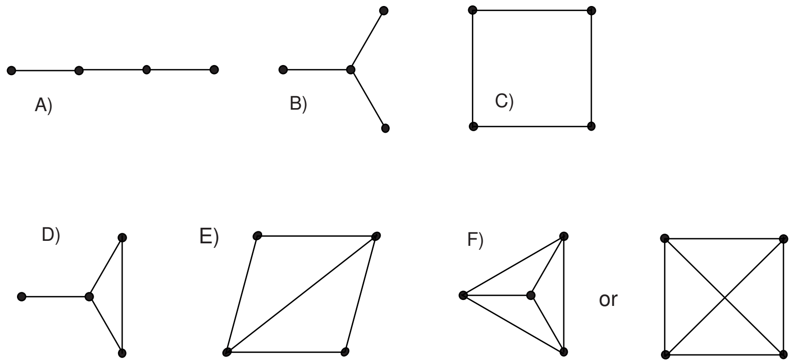

Figure 1: Possible connectivities of four critical points in the case of a quintic superpotential.

where α_i, $i = 1, \ldots, 4$ are complex deformation parameters. Critical points of the superpotential are points z_a where $dW(z_a) = 0$. The possible connections of the critical points in this case are shown in figure 1, where each dot represents a root i (or critical point) and each line represents a possible domain wall solution interpolating between critical points i and j. When such a line exists between two given roots, $|\mu_{ij}| = 1$, and it thus represents a possible BPS state. When there is no line between two given roots $\mu_{ij} = 0$ and there can be no BPS state. Therefore it is easy to see that figure 1 exhausts all possible connections of critical points. So one expects that in some domain the number of BPS states is the smallest possible, namely three (as in figures 1-A, -B), while in some other domain the maximum number of BPS states, namely six (as in figure 1-F) is obtained, depending on the choice of superpotential, i.e. deformation parameters α_i. However, figure 1-C deserves some further comments. In the following we will see that in no finite domain of deformation parameters will the connectivity be as in C. This is actually easy to understand geometrically. In such a four-gon — defined by roots i, j, k and l — are $i'j'k'$ connected for any cyclic permutation of the four roots but i' and k' are not connected. So all the angles of $i'j'k'$ has to be at least $90°$ (and actually *greater* than $90°$ for a finite domain of deformation parameters). But the sum of the angles of the 4-gon is $360°$ and we have a contradiction. What about the case of $k > 4$ number of critical points? For a k-gon, the sum of angles is $(k-2) \cdot 180°$. For a configuration with no "internal" BPS solitons the sum of angles should be at least $k \cdot 90°$. For $k \geq 5$ one might have such domains.

The soliton structure of any such massive deformation of a conformal theory is characterized by the matrix $S = 1 - A$, where A is an upper triangular matrix whose elements A_{ij} for $i < j$ are exactly $A_{ij} = \mu_{ij} = \Delta_i \circ \Delta_j$ [4]. However, this matrix does not in itself classify the possible junctions. Precisely for this reason will figure 1-F need some further comments. *If* the actual location of the critical points is as in figure 1-F (triangle), then one can obtain a junction of domain walls by occupying all six states (this junctions will look like a circle with three legs coming out). But imagine that the locations of the critical points are as in figure 1-F (square) with the inclusion of the two BPS states connecting diagonal corners. Occupying all these states would not give rise to a stable junction.

In eq. (5.2) we fix two of the critical points to be at $z = \pm 1$, so that the four critical points are located at

$$z_1 = -1\,, \qquad z_2 = 1\,, \qquad z_3 = \mu\,, \qquad z_4 = \lambda\,, \tag{5.3}$$

corresponding to the superpotential which takes the following form:

$$W = z^5 - \frac{5}{4}(\mu+\lambda)z^4 + \frac{5}{3}(\mu\lambda - 1)z^3 + \frac{5}{2}(\mu+\lambda)z^2 - 5\mu\lambda z\,. \tag{5.4}$$

We thus have two complex parameters μ and λ to vary, and in general it is not easy to visualize different domains in this space of parameters. A systematic way is to fix one of the complex parameters, say μ and have a sliced view of the separatrix walls. We will consider a few representative values of μ:

1. the case where three points $z = -1, z = 1, z = \mu$ are at vertices of an equilateral triangle. (This includes the case we already discussed in the introduction which corresponds to the situation where the fourth critical point is at the center of the triangle. For this case we already know the possible connectivities of the critical points and we can use it as the "initial data" for our analysis.)

2. the case where three points are colinear on the real axis and finally

3. the case which includes the $\mathbb{Z}_4$ symmetric case.

For a generic configuration of roots (i.e. when z_3 is not colinear with z_1 and z_2) one can obtain the complete set of separatrix curves as follows. Pick any two roots z_i, z_j $(i > j)$ and consider the basic separatrix curve joining them as defined by the equation $F_{4ij}F_{4ji}F_{j4i} = 0$. Then the condition that the product of all these groups of terms vanishes is the equation for the "complete" separatrix curve, just as it is in the case of a single pair of roots when we have a quartic superpotential. Now we will focus on the three cases. In the first case we take the three fixed roots to be at the vertices of an equilateral triangle, i.e. $z_1 = -1, z_2 = +1, z_3 = i\sqrt{3}$ (see figure 2).

In this case there is a $\mathbb{Z}_3$-symmetry generated by rotations of $2\pi/3$ in the center of the triangle. In the second case we take the roots to be colinear $z_1 = -1, z_2 = +1, z_3 = +3$ (see figure 3). In this case there is a $\mathbb{Z}_2$ symmetry generated by reflections along the vertical line $\lambda_2 = 0$. The last configuration is where $z_1 = -1, z_2 = +1$ and $z_3 = -1 + 2i$ (see figure 4) and so contains the $\mathbb{Z}_4$-symmetric superpotential (for $z_4 = 1 + 2i$) which has been much studied.

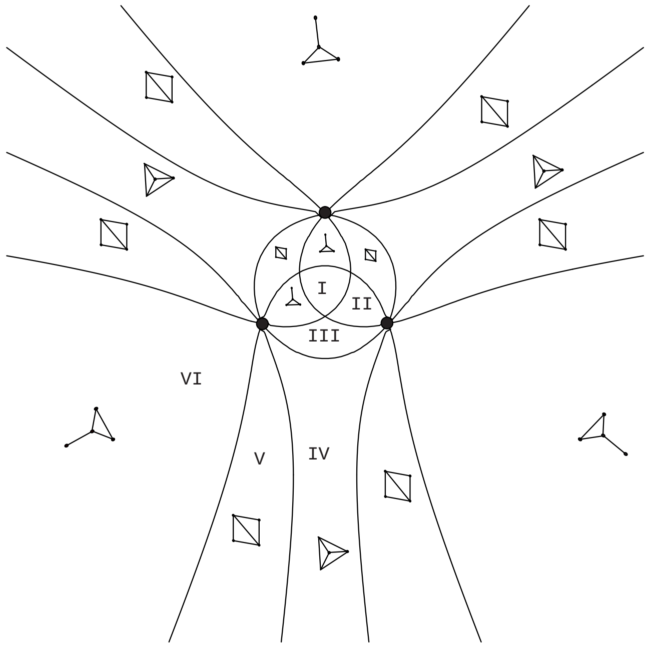

Figure 2: $\mathbb{Z}_3$-symmetric case. Connectivity of roots depending on the value of $z_4 = \lambda$. The three fixed roots are located at $z_1 = -1$, $z_2 = 1$ and $z_3 = i\sqrt{3}$.

For any fixed value $z_3 = \mu$, one can write down the separatrix equation as a sixth order equation in λ_1 and λ_2 where $\lambda = \lambda_1 + i\lambda_2$. Choosing for example the pair of roots (z_2, z_4) and (z_4, z_3) we obtain:

$$\begin{aligned}
0 = & -\frac{5}{12}\lambda_1^5\lambda_2(-1+\mu)^3(3+\mu) + \frac{5}{6}\lambda_1^4\lambda_2(-1+\mu)^3(1+3\mu+\mu^2) - \\
& -\frac{5}{12}\lambda_1^3\lambda_2(-1+\mu)^3(-5+\mu+3\mu^2+\mu^3) - \\
& -\frac{5}{18}\lambda_1^2\lambda_2(-1+\mu)^3(6+18\mu+6\mu^2+\lambda_2^2(6+3\mu+\mu^2)) - \\
& -\frac{1}{18}\lambda_2(-1+\mu)^3\Big((15\mu^2(2+\mu)+\lambda_2^4)(-3+6\mu+2\mu^2) + \\
& \qquad\qquad + 5\lambda_2^2(-2+9\mu+3\mu^2)\Big) + \\
& +\frac{5}{36}\lambda_1\lambda_2(-1+\mu)^3\Big(3\lambda_2^4(3+\mu) + 3\mu(8+9\mu+3\mu^2) + \\
& \qquad\qquad + \lambda_2^2(15+13\mu+9\mu^2+3\mu^3)\Big) .
\end{aligned} \tag{5.5}$$

The sliced view of this separatrix equation will be shown in figure 2–4 for particular values of μ mentioned above. Note that the horizontal line $\lambda_2 = 0$ is a solution of (5.5), however in crossing this line the number of BPS states does not jump and therefore it is not included in figures 2–4.

We start by counting the number of possible BPS states for the $\mathbb{Z}_3$-symmetric configuration of roots, see figure 2. Asymptotically we have 12 domains in this

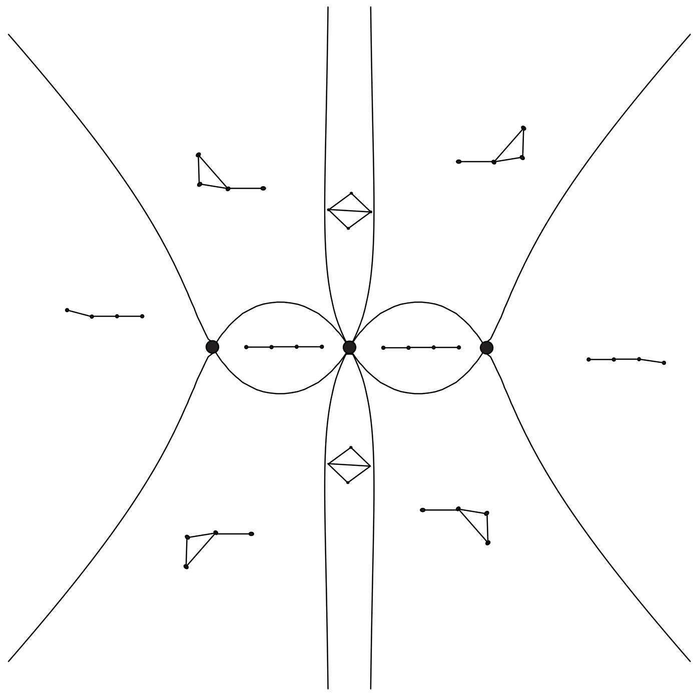

Figure 3: $\mathbb{Z}_2$-symmetric case. Connectivity of roots depending on the value of $z_4 = \lambda$. The three fixed roots are located at $z_1 = -1$,$z_2 = 1$ and $z_3 = 3$.

case and once the BPS configuration(s) is determined in four of these domains, the configurations in the other asymptotic domains follow by using $\mathbb{Z}_3$-symmetry. Generally we will call z_1 as root 1, z_2 as root 2 and so on. We start with the most symmetrical configuration, where the fourth root λ is in the center of the triangle defined by the roots 1, 2 and 3. We call this small domain *I*. *I* is defined as the intersection of three domains: one where 1 is connected to 4 and 4 is connected to 3, but 1 and 3 is *not* connected (this comes from the basic separatrix curve connecting 1 and 3), one where 2 is connected to 4 which is connected to 3, but 2 and 3 is not connected and finally one where 1 is connected to 4 and 4 is connected to 2 but 1 and 2 are not connected. This shows that the connectivity of the diagram in domain *I* must be of type B. The number of possible BPS states is therefore 3. The number of BPS states in the other domains can now be determined by using the rules described in the last section in crossing the different separatrix curves.

I$\rightarrow$II: in going to domain *II* one crosses the line $F_{143} = 0$ and since there was no connecting between 1 and 3 to start with these two roots gets connected by a BPS solution. The number of BPS states in *II* is then 4 and the connectivity is of type D. **II$\rightarrow$III**: in going to domain *III* one crosses the line $F_{243} = 0$ and since there was no connecting between 2 and 3 to start with these two roots gets connected by a BPS solution. The number of BPS states in *III* is then 5 and the connectivity is of type E. **III$\rightarrow$IV**: in going to domain *IV* one crosses the line $F_{142} = 0$ and since there

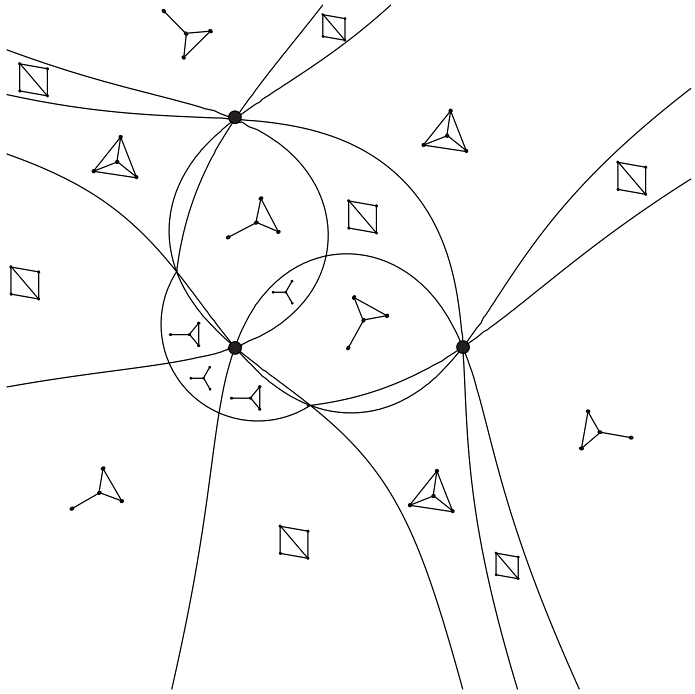

Figure 4: Connectivity of roots depending on the value of $z_4 = \lambda$. The three fixed roots are located at $z_1 = -1$, $z_2 = 1$ and $z_3 = -1 + 2i$. *Note*: In the empty domains the connectivity is of type E.

was no connecting between 1 and 2 to start with 1 and 2 to will be connected such that all roots are connected and the number of BPS states is 6. The connectivity is of type F. **IV**→**V**: in going to domain V one crosses the line $F_{314} = 0$ and since there was a connecting between 3 and 4 to start with, this domain wall disappears and instead 1 and 2 is connected. The connectivity is then of type E with 5 possible domain walls. **V**→**VI**: in going to domain VI one crosses the line $F_{124} = 0$ and since there was a connecting between 1 and 4 to start with, this domain wall disappears. The connectivity is then of type D with 4 possible BPS states. By $\mathbb{Z}_3$-symmetry this analysis determines the possible number of BPS states in all domains and hence we have a complete determination of the possible domain walls and junctions for a potential with this particular symmetry. For this class of superpotentials, the number of BPS states varies from three to six.

A similar analysis can be carried out for the $\mathbb{Z}_2$-symmetric case in figure 3. When we simply plot the corresponding eq. (5.5) for all pairs of roots then we get more lines than is shown in figure 3. However, some of these lines are lines of marginal stability, just like the real axis is for a quartic superpotential as discussed in section 5.1, and should be ignored.

However, here the three fixed roots are all colinear so the resulting diagram is very simple. To determine the possible BPS states in the different domains, one can start by taking $1 < z_4 < 3$ and real. In this case the configuration is known [33]: all roots are successively connected as shown in figure 3. The configuration in other domains is then simply determined by crossing the different curves of marginal stability. For this case the number of BPS states varies from three to five.

The case including the $\mathbb{Z}_4$-symmetric potential is presented in figure 4.[2] At first glance this figure looks very complicated. However, it has some features common with figure 2. For example, eight separatrix lines emanate from each critical point. For this choice of parameters, the number of BPS states varies from three (around the "center") to six (at the $\mathbb{Z}_4$ symmetric point for example). So all possible connectivities are realized, except the minimal case of three BPS states (figure 1-A) and figure 1-C of course.

We have found all the possible BPS states. Now let us focus on the junction configurations. As mentioned before, a triangle leads to a junction of three domain walls. If only one edge of the triangle is occupied, it is a 1/2 BPS state of a single domain wall. When all three edges are occupied, then all the tensions will be balanced and this will lead to a 1/4 BPS configuration of junctions of domain walls. More generally, one could have junctions of any number of domain walls. Here is how we can define a junction configuration in this case. First find the locations of the critical points in the W-space. The integral curves will be straight lines between two critical points i, j in this space, and will have corresponding "soliton" number μ_{ij}, which can also be zero. Next pick any number of critical points in the W-space, such that the successive connection of these points form a convex polygon. If all the edges have non-vanishing μ_{ij}, that is, if the polygon is closed then we have a non-trivial junction, and the inside of the polygon will have 1/4 supersymmetry. Each of the edges of the polygon will have 1/2 of the original supersymmetry, and only at the vertices, that is at the critical point, is all of the original supersymmetry preserved. Again, we see in this "graphical" understanding of various SUSY configurations that there is no room for 3/4 BPS states in the W-Z model [23].

If any of the μ_{ij} along the edge of the polygon is zero, then we cannot define the "inside" of the polygon and there will be no junction. We will have just domain walls with extend to infinity (in the coordinate space) and which never join. The number of preserved supercharges will be two.

6. Discussion

So far we have been discussing the possible BPS states and junctions in the W-Z model. We have summarized our result in figures 2–4 where we can easily read off the number of BPS states as well as possible BPS junction configurations for a given

[2] To keep the graphics simple, some of the domains in this figure are left empty. In these domains the connectivity is of type E.

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deformation parameter. In this way we obtain a complete picture of the possible BPS domain wall junctions for the models we consider. So what is the use of all this? First of all, we have used a method general enough to be utilized for counting BPS states for other types of superpotentials. Secondly, the BPS data of W-Z models (or those with other superpotentials) which can be obtained from higher-dimensional theories will reflect the BPS data of the original theories.

Apart from these practical things, we would also like to point out some of the possible connections to works done in the context of string compactifications and also brane configurations. Due to the relation to Calabi-Yau compactifications we can reinterpret our results as that of counting numbers of BPS D-branes wrapped around supersymmetric cycles. On top of each domain wall there is a "sphere" wrapping around a supersymmetric cycle, whose radius vanishes at the critical points. This is reminiscent of toric geometry: we have vanishing spheres at the critical points and have finite radius cycles over the line interpolating two critical points. That is spheres in the internal dimension over the domain walls will be revealing some of the structures of Calabi-Yau spaces. In particular, it has been shown that certain toric geometries, which has vanishing cycles, can be translated into a brane configuration [34]. Thus another very interesting application comes from the T-dual picture of the Calabi-Yau compactifications, i.e. the brane configurations. As an example consider the following situation. The brane configuration for $\mathcal{N}=1$ SU(N_c) supersymmetric YM is given in type-IIA string theory with N_c D4 branes extending between two sets of coincident NS5 branes as follows. With the coordinates

$$s = x^6 + ix^{10}\,, \qquad v = x^4 + ix^5\,, \qquad w = x^8 + ix^9\,, \tag{6.1}$$

the D4-brane is located at $v = w = 0$ and extended in the s-direction. The NS5-brane is located at $s = w = 0$ and is extended in the v-direction, and the NS5'-brane is at $v = 0$, $s = L$ and is extended along the w-direction. Consider a configuration of m coincident NS5 branes connected by N_c D4 branes to m' coincident NS5' branes. There will be two adjoint superfields Φ, Φ', which describe fluctuations of the fourbranes in the w and v directions, respectively, whose classical superpotential is

$$W = \frac{a}{m+1}\,\mathrm{Tr}\,\Phi^{m+1} + \frac{a'}{m'+1}\,\mathrm{Tr}\,\Phi^{m'+1} + \cdots. \tag{6.2}$$

Imagine having the k NS5 branes in the (x^8, x^9) plane at k different points w_j. Since the $\{w_j\}$ correspond to locations of heavy objects they appear as parameters rather than moduli in the gauge theory description and give rise to a polynomial superpotential for Φ where $W'(\Phi) = a\prod_{j=1}^{m}(\Phi - w_i)$. This shows how superpotentials of the form discussed in this paper can arise from brane configurations.

Another very interesting result can be obtained with similar methods in the study of BPS states of Argyres and Douglas superconformal theories [35, 36, 37], as in ref. [33]. In fact, if we consider a degenerate choice of polynomial, where

$P_m = (dW/dx)^2$, the problem becomes identical to the problems we have discussed here. Exact equations for the separatrix curves can be obtained but will be quite complicated and involve certain elliptic functions.

As discussed in section 3, when we consider type-IIA string theory compactified on a Calabi-Yau fourfold we obtain a $(1+1)$-dimensional effective theory which gives the vacuum structure and the D4 branes wrapping around the supersymmetric cycles give solitons interpolating the vacua. If we start with M-theory, which is the strong coupling regime of the type-IIA theory, we end up with an effective $(2+1)$-dimensional theory, with similar vacuum and domain wall structure. However, there is something more. Due to one more space dimension, the vacua can arrange such that there can be junctions of the domain walls. From the point of view of string theory this extra dimension is a non-perturbative effect. Thus having a full understanding of lower-dimensional integrable models might not guarantee an understanding of higher-dimensional integrable model, just as understanding fully perturbative field theory does not guarantee any insight into a fully non-perturbative field theory.

The superpotential we have studied in this paper is the simplest kind involving only one type of field. There are many extensions that can be made with multiple species of fields. One nice extension would be the study of the $D-E$ series [37] of singularities and the corresponding W-Z models. In the case of W-Z models of A_n type, one always has a single type of domain walls between two critical points, because there is only one type of complex scalar field in the theory. However, if we have multiple species of scalar fields we have the possibility of multiple types of domain walls between the critical points. It would be interesting to generalize the method used here to study these systems and also find junctions of multiple species of branes. Theories such as the CP^n models have multiple species of domain walls between critical points, which can be labeled by a group theory index. So when we consider junctions of a multiple of these domain walls, perhaps only a certain combinations will lead to a BPS junction. This certainly deserves a further study.

There are still some open questions, we would like to answer in the near future: How do we describe junctions of domain walls in the higher-dimensional Calabi-Yau geometry? Are stable junctions classified by some topological class, related to the higher-dimensional geometry?

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